MAHLER MEASURE OF POLYNOMIAL ITERATES

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Abstract. Granville recently asked how the Mahler measure behaves in the context of polynomial dynamics. For a polynomial $f(z) = z^d + \ldots \in \mathbb{C}[z],\ \deg(f) \geq 2$, we show that the Mahler measure of the iterates f^n grows geometrically fast with the degree d^n, and find the exact base of that exponential growth. This base is expressed via an integral of $\log^+ |z|$ with respect to the invariant measure of the Julia set for the polynomial f. Moreover, we give sharp estimates for such an integral when the Julia set is connected.

1. Main result

For an arbitrary polynomial $P(z) = c_n \prod_{k=1}^n (z - z_k) \in \mathbb{C}[z]$ with $c_n \neq 0$, the Mahler measure is given by

$$M(P) := \exp\left(\frac{1}{2\pi}\int \log|P(e^{i\theta})|\, d\theta\right) = |c_n| \prod_{k=1}^n \max(1, |z_k|), \tag{1}$$

where the second equality is a well known consequence of Jensen's formula, see [2], [7] and [11] for background and applications.

Let $f(z) = z^d + \ldots \in \mathbb{C}[z],\ \deg(f) \geq 2$, and consider the n-fold iterates for f denoted by f^n, which are monic polynomials of degree $d^n,\ n \in \mathbb{N}$. At a recent conference [9], Granville asked interesting questions on the behavior of the Mahler measure under composition of polynomials. In particular, how the Mahler measure of the polynomial iterates f^n behaves as $n \to \infty$. Our primary goal is to show that the Mahler measure of f^n grows geometrically fast with the degree d^n. In order to state a precise result, we need to introduce the Julia set of f denoted by J, which is a completely invariant compact set under iteration of f, see, e.g., [6] for details. It is also known that there is a unique unit Borel measure μ_J supported on J that is invariant under f. In fact, μ_J is the equilibrium measure of J in the sense of logarithmic potential theory (see [13] and [6]), and it expresses the steady state distribution of charge if J is viewed as conductor.

Theorem 1.1. *If $f(z) = z^d + \ldots \in \mathbb{C}[z],\ \deg(f) \geq 2$, is different from the monomial z^d, then we have*

$$\lim_{n\to\infty} d^{-n} \log M(f^n) = \int \log^+ |z| d\mu_J(z) > 0, \tag{2}$$

where μ_J is the invariant (equilibrium) measure of the Julia set J for f.

Remark 1.2. *If $f(z) = z^d$ then $f^n(z) = z^{d^n},\ n \in \mathbb{N}$, and $M(f^n) = 1,\ n \in \mathbb{N}$, by (1). Also note that the smallest value of $\int \log^+ |z| d\mu_J(z)$ is 0 that is attained for $f(z) = z^d$ with $J = \mathbb{T} := \{|z| = 1\}$ and $d\mu_{\mathbb{T}}(e^{it}) = dt/(2\pi),\ t \in [0, 2\pi)$.*

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In light of (2), we arrive at the question: How large can $\int \log^+ |z| d\mu_J(z)$ be? Since the location of the Julia set J vary with f in such a way that J can be essentially anywhere in the complex plane, the value of this integral can be arbitrarily large with the values of $\log^+ |z|$. Indeed, if $J \subset \{z : |z| > R\}$ then $\int \log^+ |z| d\mu_J(z) \ge \log R$ because μ_J is the unit measure, where $R > 1$ can be arbitrarily large. However, if we make proper normalization assumptions, then we obtain some precise upper bounds stated below.

Let K be the filled-in Julia set that consists of the Julia set J and the union of the bounded components of its complement $\mathbb{C} \setminus J$, see [6, p. 65]. It is clear that $J = \partial K$, so that K is connected if and only if J is connected, which is known to hold if and only if all the critical points of f are contained in K, see [6, p. 66]. Moreover, J and K share the same equilibrium measure $\mu_J = \mu_K$, cf. [3] and [13].

Theorem 1.3. *If $f(z) = z^d + \ldots \in \mathbb{C}[z]$, $\deg(f) \ge 2$, J is connected and $0 \in K$, then*

$$\int \log^+ |z| d\mu_J(z) \le \int_1^4 \frac{\log t \, dt}{\pi \sqrt{t(4-t)}} \approx 0.6461318945. \tag{3}$$

Equality holds above for $J = K = [0, 4]$ and $f(z) = 2\, T_d(z/2-1)$, where $T_d(z) = \cos(d \arccos z)$ is the classical Chebyshev polynomial.

Symmetry assumptions also produce interesting results such as the one below.

Theorem 1.4. *If $f(z) = z^d + \ldots \in \mathbb{C}[z]$, $\deg(f) \ge 2$, is either an odd or an even function, and J is connected, then*

$$\int \log^+ |z| d\mu_J(z) \le 2 \int_1^2 \frac{\log t \, dt}{\pi \sqrt{1-t^2}} \approx 0.3230659472. \tag{4}$$

Equality holds above for $J = [-2, 2]$ and $f(z) = 2\, T_d(z/2)$, where $T_d(z) = \cos(d \arccos z)$.

A classical example that satisfies the assumptions of Theorem 1.4 is given by the family of quadratic polynomials $f(z) = z^2 + c$ with c from the Mandelbrot set, see Chapter VIII of [6].

We remark that the growth of the Mahler measure for the iterates exhibited here is essentially due to the intrinsic connection of the Mahler measure to the unit circle. A more suitable version of the Mahler measure for the dynamical setting is known, see the recent papers [5] and [4], where the last one surveys many developments in the area. Another related notion is dynamical (or canonical) height, see [14] for a comprehensive exposition. There are many other connections of the Mahler measure and its generalizations with polynomial dynamics. Thus the integral of (2) can be interpreted as the Arakelov-Zhang pairing of f and z^2 that arises as a limit of average Weil heights in [12]. It is practically impossible to discuss all these interesting relations in detail in this short note.

For the proofs of Theorems 1.1, 1.3 and 1.4, we need the well known result of Brolin [3, Theorem 16.1] on the equidistribution of preimages for the iterates f^n:

Brolin's Theorem. *Let $w \in \mathbb{C}$ be any point with one possible exception. Consider the preimages of w under f^n denoted by $\{z_{k,n}\}_{k=1}^{d^n}$, i.e., all solutions of the equation $f^n(z) = w$ listed according to multiplicities. Define the normalized counting measures in those preimages* *by*

$$\tau_n := \frac{1}{d^n} \sum_{k=1}^{d^n} \delta_{z_{k,n}}, \tag{5}$$

where δ_z denotes a unit point mass at z. Then we have the following weak convergence:*

$$\tau_n \xrightarrow{*} \mu_J \quad \text{as } n \to \infty. \tag{6}$$

Brolin's result has the following implication, which is crucial for our purposes.

Corollary 1.5. *If $f(z) = z^d + \ldots \in \mathbb{C}[z]$, $\deg(f) \geq 2$, is not the monomial z^d, then we have for the zeros of f^n denoted by $\{z_{k,n}\}_{k=1}^{d^n}$ that*

$$\tau_n = \frac{1}{d^n} \sum_{k=1}^{d^n} \delta_{z_{k,n}} \xrightarrow{*} \mu_J \quad \textit{as } n \to \infty. \tag{7}$$

Proof. The exceptional points in Brolin's Theorem arise as values omitted by the family of iterates $\{f^n\}_{n=1}^{\infty}$ in a neighborhood of any point $\zeta \in J$. It follows that there are at most two such omitted values by Montel's theorem on normal families, for otherwise the family $\{f^n\}_{n=1}^{\infty}$ would be normal in that neighborhood, which contradicts the definition of the Julia set J for f. Moreover, Lemma 2.2 of [3] states that the exceptional values are the same for all points $\zeta \in J$. Since f is a polynomial in our settings, it certainly omits the value ∞ in every disk $\{z : |z - \zeta| < r\}$, where $r > 0$, $\zeta \in J$, so that at most one exceptional value can occur in this case. For example, if $f(z) = z^d$ then this exceptional value is 0 in every disk $\{z : |z - \zeta| < 1\}$, where $\zeta \in J = \mathbb{T}$ the unit circumference. However, 0 cannot be an exceptional value for any polynomial in Theorem 1.1. Indeed, since $\deg(f) \geq 2$ and f is not the monomial z^d, there is a root $w_0 \neq 0$ of f. If we assume that 0 is an exceptional point for Brolin's Theorem, equivalently an omitted value for the family $\{f^n\}_{n=1}^{\infty}$ in a neighborhood V of a point $\zeta \in J$, then the same must be true for w_0 because $f^n(z_0) = w_0$ for a point $z_0 \in V$ implies $f^{n+1}(z_0) = 0$. But two finite omitted values $0, w_0$ mean that the family $\{f^n\}_{n=1}^{\infty}$ must be normal in V, contradicting the definition of the Julia set J. Thus 0 is not an exceptional point, and Corollary 1.5 is an immediate consequence of Brolin's Theorem. □

2. Proofs

We continue with the same notations as before.

Proof of Theorem 1.1. It is clear from (1) that

$$d^{-n} \log M(f^n) = \frac{1}{d^n} \sum_{k=1}^{d^n} \log^+ |z_{k,n}| = \int \log^+ |z| \, d\tau_n(z).$$

Since $\log^+ |z|$ is a continuous function in $\mathbb{C}$, the limit relation in (2) follows from the weak* convergence of (7). One only needs to observe here that the sets $\{z_{k,n}\}_{k=1}^{d^n}$ are uniformly bounded for all $n \in \mathbb{N}$, say belong to a fixed disk $D_R = \{z : |z| \leq R\}$, so that $\log^+ |z|$ can be extended from D_R to $\mathbb{C} \setminus D_R$ as a continuous function with compact support in $\mathbb{C}$.

The inequality in (2) follows from the work of Fernández [8], who showed that the Julia set J of f different from z^d must have points in the domain $\Delta = \{z : |z| > 1\}$. It is well known that $\operatorname{supp}\mu_J = J$, see [3, Lemma 15.2] and [13, pp. 195–197]. Thus

$$\int \log^+ |z| d\mu_J(z) = \int_\Delta \log |z| d\mu_J(z) > 0.$$

□

Proof of Theorem 1.3. Recall that the logarithmic capacity of the Julia set for a monic polynomial is equal to 1, see Lemma 15.1 of [3] and Theorem 6.5.1 of [13] for a detailed proof. The book [13] contains a complete account on logarithmic potential theory, and on capacity in particular. Since $J = \partial K$, the equilibrium measure of K is $\mu_K = \mu_J$, and the capacity of K is 1, cf. [13]. Clearly, K is a connected set because J is so. The conditions that the capacity of K is 1, $0 \in K$ and K is connected, introduce restrictions on the size of K and, consequently, on the size of the integral $\int \log^+ |z| d\mu_J(z)$ in (2). Theorem 6.2 of [1], see also Corollary 6 of [10], gives that the largest value of this integral is attained when $K = [0, 4] = J$, in which case it is well known [13] that

$$d\mu_K(x) = d\mu_J(x) = \frac{dx}{\pi\sqrt{x(4-x)}}, \quad x \in (0, 4).$$

To apply Theorem 6.2 of [1], we also need to note that $\log^+ |z| = \max(0, \log |z|)$ is clearly a convex function of $\log |z|$. Thus we have the upper bound (3)

$$\int \log^+ |z| d\mu_J(z) \le \int_1^4 \frac{\log t\, dt}{\pi\sqrt{t(4-t)}} \approx 0.6461318945.$$

The case of equality for $J = [0, 4]$ is attained by the polynomial $f(z) = 2\,T_d(z/2 - 1)$, where $T_d(z) = \cos(d \arccos z)$ is the classical Chebyshev polynomial of the first kind, see Sections 1.6.2 and 6.2 of [14] for details. □

Proof of Theorem 1.4. We proceed with a proof similar to the previous one, but use Corollary 6.3 of [1] instead of Theorem 6.2 of [1]. We have that capacity of J is 1 by Theorem 6.5.1 of [13], and J is connected by our assumption. Corollary 6.3 of [1] is applied to the filled-in Julia set K, so that $J = \partial K$, where the equilibrium measure of K is $\mu_K = \mu_J$, and the capacity of K is 1. Again, K is connected because J is so. Moreover, both J and K are symmetric with respect to the origin because f is even or odd. If f is odd, then 0 is a fixed point of f, implying that $0 \in K$. If f is even, then 0 is a critical point of f, hence $0 \in K$ because we assume that J is connected (cf. [6, p. 66]). Thus $0 \in K$ under our assumptions, and we obtain from Corollary 6.3 of [1] that the largest value of the integral in (4) is attained for $J = K = [-2, 2]$:

$$\int \log^+ |z| d\mu_J(z) = \int \log^+ |z| d\mu_K(z) \le 2 \int_1^2 \frac{\log t\, dt}{\pi\sqrt{1-t^2}} \approx 0.3230659472,$$

where we used that the equilibrium measure for $J = K = [-2, 2]$ is the Chebyshev distribution [13]

$$d\mu_K(x) = d\mu_J(x) = \frac{dx}{\pi\sqrt{4-x^2}}, \quad x \in (-2, 2).$$

It is well known that $J = [-2, 2]$ for $f(z) = 2\,T_d(z/2)$, where $T_d(z) = \cos(d \arccos z)$, see Sections 1.6.2 and 6.2 of [14]. □

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References

[1] A. Baernstein II, R. S. Laugesen, and I. E. Pritsker, Moment inequalities for equilibrium measures in the plane, Pure Appl. Math. Q. 7 (2011), 51–86.

[2] P. Borwein, Computational Excursions in Analysis and Number Theory, Springer-Verlag, New York, 2002.

[3] H. Brolin, Invariant sets under iteration of rational functions, Ark. Mat. 6 (1965), 103–144.

[4] A. Carter, M. Lalín, M. Manes, and A. Miller, Dynamical Mahler Measure: A survey and some recent results, arXiv:2204.04101.

[5] A. Carter, M. Lalín, M. Manes, A. Miller, and L. Mocz, Two-variable polynomials with dynamical Mahler measure zero, Res. Number Theory 8 (2022), Paper No. 25.

[6] L. Carleson and Th. W. Gamelin, Complex Dynamics, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.

[7] G. Everest and T. Ward, Heights of Polynomials and Entropy in Algebraic Dynamics, Springer-Verlag, London, 1999.

[8] J. L. Fernández, A note on the Julia set of polynomials, Complex Variables 12 (1989), 83–85.

[9] A. Granville, Problem posed at the conference "Equidistribution and Arithmetic Dynamics," Oklahoma State University, June 20-24, 2022.

[10] R. Laugesen, Extremal problems involving logarithmic and Green capacity, Duke Math. J. 70 (1993), 445–480.

[11] J. McKee and C. Smyth, Around the Unit Circle: Mahler Measure, Integer Matrices and Roots of Unity, Springer, 2021.

[12] C. Petsche, L. Szpiro, and T. J. Tucker, A dynamical pairing between two rational maps, Trans. Amer. Math. Soc. 364 (2012), 1687-1710.

[13] T. Ransford, Potential Theory in the Complex Plane, Cambridge University Press, Cambridge, 1995.

[14] J. H. Silverman, The Arithmetic of Dynamical Systems, Springer, 2007.

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